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Study of a Self-Consistent Calculation of the Ω^- as a $\bar{K}\Xi$ Bound State*

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The $\bar{K}\Xi\Omega^-$ system has been studied in a self-consistent manner, both to see if the Ω^- can be interpreted as a $T=0\bar{K}\Xi$ bound state, and to study such calculations in a situation where only one two-particle channel need be considered. We follow closely the method of Singh and Udgaonkar, and Pati, using for dynamical singularities both a short cut due to single particle exchange and a self-consistently determined distant left cut. Only the $J=\frac{3}{2}^+$ partial wave is considered. The main result is a curve relating the strength (called g^2) of the single-particle exchange-forces to the position of a bound or resonant state. The binding energy is found to increase with increasing g^2 until the mass of the bound state is about 1680 MeV. Further increases in g^2 have essentially no effect. On the other hand, it is found that resonant states exist for $g^2 \leq 0$, so that the calculation as it stands allows resonances with repulsive single-particle exchange forces. The calculation in its present form is consistent with bound or resonant states for both $T=0$ and $T=1$, the former at about 1680 MeV for reasonable values of g^2. It is shown that the results are essentially independent of subtraction point, matching points, and similar parameters. The dependence of the results on the short cut and on the distant cut is discussed. The manner in which calculations using only the Born approximation for the dynamical singularities can give the same results as ours is indicated. The possible existence of two zeros in Re D, noticed by Abers and Zachariasen, is briefly discussed. A second zero does appear in the present calculation at about 4 GeV., but results from the phase shift decreasing through $\frac{1}{2}\pi$ rather than from the Abers-Zachariasen mechanism. Thus, no alternative solution is found in the present calculation.

I. INTRODUCTION

THE $\bar{K}\Xi$ system, with strangenesss -3, is of interest for two reasons. First, the recently discovered[1] Ω^-, with $Y=-2$, $T=0$, and $J=\frac{3}{2}^+$, may perhaps be considered dynamically as a $T=0K\Xi$ bound state in the $p_{3/2}$ partial wave. Second, it is a physical system strongly coupled to no other two-particle baryon-meson system, so that the results of a single-channel calculation are of particular interest. In the following we examine the $\bar{K}\Xi\Omega^-$ interaction in a self-consistent manner for both of these reasons.

The present calculation is a self-consistent one in the following sense. Values for the position and residue of the Ω^- pole in the $p_{3/2}$ amplitude are guessed initially and used as input. Values for the same quantities are then obtained as output from an N/D solution. The new values are used as input, leading to new output values, and the procedure is repeated (by a computer) until the output and input values are the same, if possible. It is not a true bootstrap calculation because the left-hand singularities are fixed input information (although the fixed parameters are varied over a wide range to obtain some idea of their effect). Such a bootstrap calculation is in principle possible since (for example) the Σ is perhaps partially a $K\Xi$ bound state held together by Ω^- exchange, just as the Ω^- is perhaps a $\bar{K}\Xi$ bound state held together primarily by Σ exchange. Since $K\Xi$ is coupled to $\pi\Sigma$, $\pi\Lambda$, and $\bar{K}N$, this is an interesting multichannel problem which we do not consider further.

The $p_{3/2}$ partial-wave scattering amplitude $g_{1+}=N/D$ is characterized, from our point of view, by its singularities. It has a unitarity cut, built into D, and any bound states will appear as zeros of D. The dynamical singularities are those contained in N.

In the present calculation we follow Singh and Udgaonkar[2] (SU) and Pati[3] (P) and take N to contain a nearby short cut due to single-particle exchange, and a distant cut. The latter is characterized by parameters which are self-consistently determined. In the more traditional bootstrap calculation one sets N equal to the amplitude for single-particle exchange. Although the two types of calculations give quite similar results, from some points of view they appear to be quite different. Their relation is discussed in some detail within the framework of our results. We also examine the problem, recently noted by Abers and Zachariasen,[4] of two alternative solutions to our self-consistent calculation.

Section II summarizes the details of the calculation. The main results are given in Sec. III (Table I and Fig. 1), and are discussed, along with the topics mentioned above, in Sec. IV.

II. DETAILS OF THE CALCULATION

A calculation of this type has been carried out by Singh and Udgaonkar[2] for the N^* and by Pati[3] for the Ξ^*. Our treatment follows that of Pati quite closely; the reader is referred to his paper for additional details. For completeness we summarize the procedure to be followed. It is assumed throughout that the Ω^- has $J=\frac{3}{2}^+$, so only the $p_{3/2}$ partial-wave amplitude will be considered.

* Supported by the National Science Foundation.

[1] V. E. Barnes, P. L. Connolly, D. J. Crennell, B. B. Culuick, W. C. Delaneg *et al.*, Phys. Rev. Letters, **12**, 204 (1964).

[2] V. Singh and B. M. Udgaonkar, Phys. Rev. **130**, 1177 (1963). We refer to this paper in the following as SU.

[3] J. C. Pati, Phys. Rev. **134**, B387 (1964). We refer to this paper in the following as P.

[4] E. Abers and F. Zachariasen, California Institute of Technology (to be published).

The kinematics for baryon-meson scattering has been given in detail in the literature.[5] We consider the process

$$\bar{K}(q_1)+\Xi(p_1)\to\bar{K}(q_2)+\Xi(p_2). \quad (1)$$

The four-momenta of the particles are given in the parentheses. Defining the Mandelstam variables s, t, and u by

$$s=-(p_1+q_1)^2,$$
$$t=-(q_2-q_1)^2,$$
$$u=-(p_2-q_1)^2,$$
$$s+t+u=2m^2+2\mu^2,$$

where m is the Ξ mass and μ the K mass, the baryon-meson scattering amplitude has the form

$$T=-A(s,t,u)+i\gamma\cdot QB(s,t,u),$$

where $Q=\frac{1}{2}(q_1+q_2)$. The magnitude of the center-of-mass three-momentum is given by

$$q^2=[s-(m+\mu)^2][s-(m-\mu)^2]/4s. \quad (2)$$

$W=s^{1/2}$ is the total energy in the center-of-mass system, and with θ the center-of-mass scattering angle, $t=-2q^2(1-\cos\theta)$.

Following SU and P we use the partial-wave amplitude

$$g_{1+}=se^{i\delta_{1+}}\sin\delta_{1+}/q^3 \quad (3)$$
$$=\{[(W+m)^2-\mu^2][A_1+(W-m)B_1]+[(W-m)^2-\mu^2][-A_2+(W+m)B_2]\}/16\pi q^2, \quad (4)$$

where

$$(A_l,B_l)=\frac{1}{2}\int_{-1}^{1}d(\cos\theta)P_l(\cos\theta)(A(s,t,u),B(s,t,u)). \quad (5)$$

The positions of the singularities of a general partial-wave amplitude have been given by Kennedy and Spearman,[6] and for this particular amplitude they have been considered in detail in the complex W plane by Frautschi and Walecka.[5] Discussions of the singularities and approximations to them for the s plane are given in SU and P. In particular, singularities arising from single-baryon (Σ or Λ in our problem) exchange, which contributes terms of the form $g^2/(m_\Sigma^2-u)$ to the invariant amplitudes, have been retained. If the Λ and Σ masses were the same then g_{1+}^{Λ} and g_{1+}^{Σ} would be identical, except for the coupling constant factor. The sum of the two would be the same as g_{1+}^{Σ} except for the value of the coupling constant. We will work in this approximation and thus retain only g_{1+}^{Σ}, with a coupling constant g^2 which characterizes the strength of the Born approximation. We will always mean this more general case when we refer to Σ exchange. The value of g^2 will be discussed later.

[5] See, for example, S. C. Frautschi and J. D. Walecka, Phys. Rev. **120**, 1486 (1960).

[6] J. Kennedy and T. D. Spearman, Phys. Rev. **126**, 1596 (1962).

The partial-wave amplitude then contains the term

$$g_{1+}^{\Sigma}(s)=g^2\{[(W+m)^2-\mu^2](W-m_\Sigma)Q_1(a)+[(W-m)^2-\mu^2](W+m_\Sigma)Q_2(a)\}/8q^4, \quad (6)$$

where

$$a=1+(2m^2+2\mu^2-m_\Sigma^2-s)/2q^2, \quad (7)$$

and in our notation $g_{\pi N}^2\approx 15$ (so that a factor $1/4\pi$ has been absorbed into the coupling constant). This gives a cut from

$$s=L_1=(m^2-\mu^2)^2/m_\Sigma^2\approx 80m_\pi^2$$

to

$$s=L_2=2(m^2+\mu^2)-m_\Sigma^2\approx 130m_\pi^2,$$

and a cut for $-\infty<s\leqslant 0$. The short cut from L_1 to L_2 is explicitly retained. The cut for negative s is replaced by two Balázs-type poles,[7] with fixed positions but with residues to be determined. The procedure for doing this is described in detail by Pati.

The exchange of higher mass states in the u channel, and t-channel singularities, contributes other unphysical cuts in the right half-plane, all of which we neglect, and cuts for $s\leqslant 0$. The latter may be considered to be contained in the Balázs poles.

The partial-wave amplitude is written in the form

$$g_{1+}(s)=N(s)/D(s), \quad (8)$$

where, as usual, N contains all the unphysical singularities and D contains the right-hand cut. We write, ignoring inelastic processes,

$$D(s)=1-\frac{s-s_0}{\pi}\int_{(m+\mu)^2}^{\infty}ds'\frac{q'^3N(s')}{s'(s'-s)(s'-s_0)} \quad (9)$$

and

$$N(s)=b_3/(s+s_3)+b_4/(s+s_4)+N_n(s), \quad (10)$$

where

$$N_n(s)=\frac{1}{\pi}\int_{L_1}^{L_2}ds'\frac{\mathrm{Im}g_{1+}^{\Sigma}(s')D(s')}{s'-s}. \quad (11)$$

The first two terms are the Balázs poles, with $s_3=18m_\pi^2$ and $s_4=720m_\pi^2$. The determination of b_3 and b_4 will be discussed below.

$\mathrm{Im}g_{1+}^{\Sigma}(s)$ is obtained from Eq. (6) to be

$$\mathrm{Im}g_{1+}^{\Sigma}(s)=-\pi g^2\{[(W+m)^2-\mu^2](W-m_\Sigma)P_1(a)+[(W-m)^2-\mu^2](W+m_\Sigma)P_2(a)\}/16q^4. \quad (12)$$

To avoid solving coupled integral equations we follow Pati and in Eq. (11) for $N(s)$ we write

$$D(s')=1-(s'-s_0)/(s_R-s_0),$$

taking advantage of the knowledge that $D(s_R)=0$ and $D(s_0)=1$. This gives

$$N_n(s)=[(s_R-s)J(s)-J_0]/(s_R-s_0), \quad (13)$$

[7] L. A. P. Balázs, Phys. Rev. **126**, 1220 (1962).

where

$$J_0=\frac{1}{\pi}\int_{L_1}^{L_2}\mathrm{Im}g_{1+}(s')ds',\tag{14}$$

$$J(s)=\frac{1}{\pi}\int_{L_1}^{L_2}\frac{\mathrm{Im}g_{1+}(s')}{s'-s}ds'.\tag{15}$$

Writing

$$\frac{1}{s'-s}=-\frac{1}{s}\left(1+\frac{s'}{s}\right)+\frac{s'^2}{s^2(s'-s)},$$

we note that

$$N_n(s)\to \mathrm{const}/s,\tag{16}$$

for large s. Thus, following Pati, we expect that for $s\geq(m+\mu)^2\approx 168m_\pi^2$ we can write

$$N_n(s)=(b_1/(s-s_1))+(b_2/(s-s_2)).\tag{17}$$

Such a procedure is a convenient one, and it turns out that one can indeed find b_1 and b_2, depending on s_R and s_0 but not on s, for which Eq. (17), with $s_1=90m_\pi^2$, $s_2=125m_\pi^2$, gives approximately the same results as Eq. (13) for N_n if $s>(m+\mu)^2\approx 168m_\pi^2$. Below threshold we can use Eq. (13) for N_n.

We then substitute $N(s)$ from Eqs. (10) and (17) in Eq. (9) for $D(s)$, obtaining

$$D(s)=1-((s-s_0)/\pi)\sum_{i=1}^{4}b_iF(s,s_i,s_0),\tag{18}$$

where

$$F(s,s_a,s_b)=\int_{(m+\mu)^2}^{\infty}ds'\frac{q'^3}{s'(s'-s)(s'-s_a)(s'-s_b)}.\tag{19}$$

We may also write

$$N_f(s)=b_3/(s+s_3)+b_4/(s+s_4)$$

and

$$D_{f,n}(s)=\frac{s-s_0}{\pi}\int_{(m+\mu)^2}^{\infty}ds'\frac{q'^3N_{f,n}(s')}{s'(s'-s)(s'-s_0)}.$$

Finally one can use Eq. (18) in Eq. (11) to obtain an improved expression for $N_n(s)$.

To find b_3 and b_4 we follow the procedure of SU and P, equating $g_{1+}(s)$ as given by Eq. (8), with $g_{1+}(s)$ as given by Eq. (4). The invariant amplitudes in the latter are given by a fixed energy dispersion relation,

$$B(s,t,u)=\frac{R_\Sigma}{m_\Sigma^2-u}+\frac{R_\Omega}{m_\Omega^2-s}+\cdots+\frac{1}{\pi}\int_{(m+\mu)^2}^{\infty}du'\frac{\mathrm{Im}B_u(u',s)}{u'-u}.\tag{20}$$

The dots represent other terms which might be included, for example Λ and Y_1^* exchange in the u channel, and t-channel singularities in general, particularly vector-meson exchange-pole terms. Note that the Ω^- can only appear in the direct channel (pole in s) because of its strangeness. A similar expression holds for $A(s,t,u)$.

Again following SU and P, we retain only the first two terms in Eq. (20). We note that (as emphasized in SU) the fixed-energy dispersion relations will only be used at a few discrete values of s. For a given $R_\Sigma(g^2)$, if the $p_{3/2}$ projections of terms neglected in Eq. (20) are small compared to the $p_{3/2}$ projection of $R_\Sigma/(m^2-u)$, at only the few points in s where Eq. (20) is used, then the neglect of such terms is justified. Their variation with energy and the location of their singularities would not be relevant. No further consideration will be given to this approximation.

Thus, we take

$$g_{1+}(s)=g_{1+}{}^{\Sigma}(s)+g_{1+}{}^{\Omega}(s),\tag{21}$$

where $g_{1+}{}^{\Sigma}(s)$ is given by Eq. (6), and

$$g_{1+}{}^{\Omega}(s)=\frac{\gamma^2[(W+m)^2-\mu^2]}{(W_R-W)[(W_R+m)^2-\mu^2]}.\tag{22}$$

γ^2 is the residue of $g_{1+}{}^{\Omega}$ at $s=s_R=m_\Omega^2$ and is proportional to the square of the $\Xi K\Omega$ coupling constant.

Equating the two forms for g_{1+} we then have

$$[b_3/(s+s_3)+b_4/(s+s_4)+N_n(s)]\Big/\left(1-\frac{s-s_0}{\pi}\sum_{i=1}^{4}b_iF(s,s_i,s_0)\right)=g_{1+}{}^{\Sigma}(s)+\frac{\gamma^2[(W+m)^2-\mu^2]}{(W_R-W)[(W_R-m)^2-\mu^2]}.\tag{23}$$

We now use Eq. (23) at two values of s, s_{m1} and s_{m2} (and so it need not be correct except at s_{m_1} and s_{m_2}), to give two equations from which we can determine b_3 and b_4 in terms of γ^2 and W_R.

A computer program was written to accomplish the following procedure: For a given g^2, s_0, s_{m_1}, and s_{m_2},

(i) Choose a γ^2 and a $W_{R\mathrm{in}}$,
(ii) Write Eq. (23) at $s=s_{m1}$ and again at $s=s_{m2}$,
(iii) Solve the resulting equations for b_3 and b_4. [Now $D(s)$ is completely known.],
(iv) Find the value of s_R for which

$$\mathrm{Re}D(s_R)=0.\tag{24}$$

(v) Find γ^2 from

$$\gamma^2=-(1/2W_R)(N(s_R)/\mathrm{Re}D'(s_R)).\tag{25}$$

(vi) Repeat steps (i)–(v), using γ^2 and s_R as input in step (i), until $\gamma^2=\gamma_{\mathrm{in}}^2$ and $s_R=s_{R\mathrm{in}}$($s_{R\mathrm{in}}$ and γ_{in}^2 being the output from the immediately previous iteration) to within some preassigned accuracy, if possible.

(vii) Repeat steps (i)–(vi) for various initial guesses

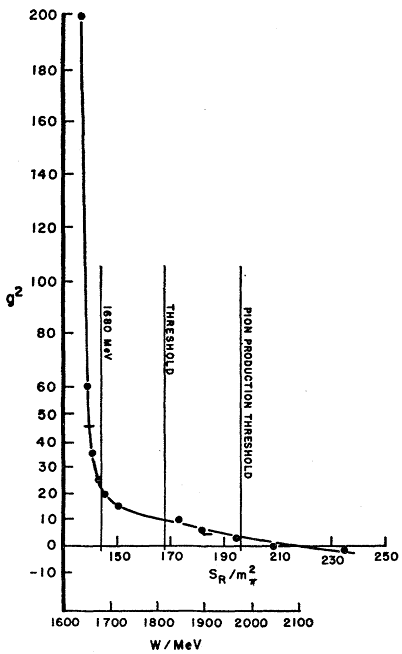

FIG. 1. Strength of exchange forces versus position of bound or resonant state, for $s_{m1}=75\,m_\pi^2$, $s_{m2}=135\ m_\pi^2$, and $s_0=50\ m_\pi^2$.

of $\gamma_{\rm in}^2$ and $s_{R\rm in}$, for various choices of s_0, s_{m1}, and s_{m2}, and for various values of g^2.

It should be noted that n parameters (such as subtraction constants or additional left-hand-cut parameters) could in principle be determined self-consistently by solving n equations from n matching points.

III. RESULTS

The results of the above procedure are presented in Table I and in Figs. 1 and 2.

Table I gives an indication of the manner in which the self-consistent results for γ^2 and s_R depend on the matching points, the subtraction point, the initial guesses for γ^2 and s_R, and the degree of self-consistency Δ. A set of values for γ^2 and s_R is called self-consistent whenever, for a given iteration, both

$$|s_{R\rm in}-s_R|<\Delta$$

and

$$|\gamma_{\rm in}^2-\gamma^2|<\Delta.$$

It was found that the derivative of Re$D(s)$ vanished at approximately the same value of s $[s\approx(350\pm10)m_\pi^2]$ for every stable, self-consistent solution, irrespective of the values of the parameters in Table I. The column in Table I headed D_m gives the value of ReD at its extremum, a positive value being a maximum and a negative value a minimum. Whether ReD had a maximum or a minimum, and its value at the extremum, depended quite strongly on the other parameters in Table I. Furthermore, some of the sets of parameters did not give rise to self-consistent solutions (e.g., for sufficiently negative g^2) and for some of these the position of the extremum would change (e.g., to $s\approx 220m_\pi^2$). The significance of these observations is not clear, but the values in the column D_m are useful for discussing the behavior of ReD. Note in particular the dependence of D_m on g^2 when all other parameters are fixed.

From Table I one can deduce that the self-consistent results are essentially independent of the matching points, the subtraction point, and the initial guesses for γ^2 and s_R. A certain amount of care must be used, however, as some combinations of parameters may lead to trouble for apparently accidental reasons. For example, with both matching points between the short cut and threshold and $g^2\sim45$, we were unable to obtain

TABLE I. Some self-consistent solutions. The units for s are m_π^2 and the coupling constants are dimensionless.

s_{m1}	s_{m2}	s_0	$\gamma_{\rm in}^2$	$s_{R\rm in}$	Δ	g^2	γ^2	s_R	D_m
75	135	50	10	168	0.5	45	11.3	140.4	−2.4
75	135	100	10	168	0.5	45	11.4	139.9	−4.8
75	135	150	10	140	0.1	45	12.0	136.9	+12.2
75	160	50	11	140	0.5	45	11.3	140.4	−2.4
75	135	50	0	168	0.1	45	11.7	137.8	−2.5
75	135	50	10	168	0.5	15	9.4	151.2	−1.9
75	160	150	11	140	0.5	15	9.5	151.0	−135.0
75	135	50	10	168	0.5	5	6.6	185.2	−0.8
75	135	100	10	168	0.5	5	6.5	183.9	−1.1
75	135	150	10	168	0.5	5	6.3	181.3	−2.4
132	165	50	6.5	184	0.1	5	7.0	182.1	−0.8
75	135	50	10	140	0.1	200	14.2	136.6	−2.6
75	135	50	10	168	0.1	0	8.4	208.5	−0.3
75	135	50	10	168	0.1	−1	11.5	234.1	−0.2
75	135	50	10	168	0.1	45	−8.9	840.2	−3.0
75	135	50	10	700	0.1	45	−8.9	840.2	−3.0
75	135	50	−10	700	0.1	30	−9.1	812.2	−2.3
75	135	50	−10	700	0.1	60	−8.8	854.1	−3.5
75	135	50	−10	700	0.1	100	−8.7	870.4	−4.2

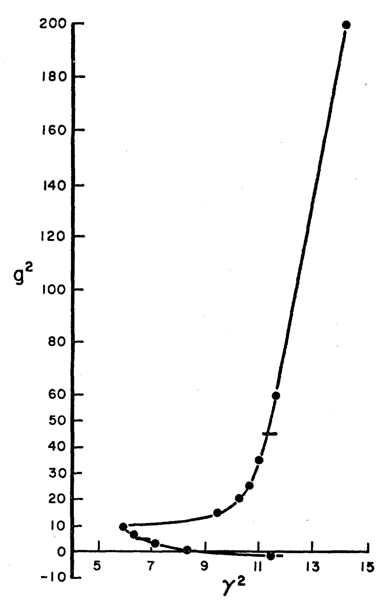

FIG. 2. Strength of exchange forces versus residue of bound or resonant state, for $s_{m1}=75\,m_\pi^2$, $s_{m2}=135\ m_\pi^2$, and $s_0=50\ m_\pi^2$.

a self-consistent solution. For $g^2=5$ the same parameters gave no difficulty.

The last solution in the first section, with $\gamma_{in}^2=0$, is an amusing one, with g_{1+}^{Ω} initially absent from Eq. (23).

In Figs. 1 and 2 we show the self-consistent results for s_R and γ^2 for $s_{m1}=75m_\pi^2$, $s_{m2}=135m_\pi^2$, and $s_0=50m_\pi^2$, for a wide range of g^2. The dots are actual results; some are also shown in Table I. The smooth curves are drawn for ease of visualization. The horitontal bars at $g^2=5$ and $g^2=45$ show the spread in output for changes in s_{m1}, s_{m2}, s_0, and Δ. The particular shape of the g^2 versus s_R curve is discussed in detail in the following. We note here, however, that self-consistent solutions exist for $g^2=0$ (no single-particle exchange) and $g^2<0$ (single-particle-exchange repulsive). For $g^2\leqslant-5$ no self-consistent solutions exist. We also point out that as g^2 increases from 45 to 200 the binding energy increases by only a few percent. One should notice, finally, that self-consistent solutions with $\gamma^2\approx0$ do not occur for any g^2.

To make an estimate of the region of validity of our results we note two criteria which have in the past been used, and which give approximately the same result. First, inelastic effects should become important near the pion production threshold, which is at $s\approx196m_\pi^2$. Second, Martin and Wali[8] chose to allow a zero of ReD imply a resonance or a bound state only for s roughly in the region from the short cut to the peaks of the principal-value integrals, $130m_\pi^2\lesssim s\leqslant190m_\pi^2$ in our case. Thus results with $s_R\gtrsim190m_\pi^2$ are to be interpreted with caution. In particular, we note that the self-consistent solutions for $g^2\lesssim5$ are outside this region.

IV. DISCUSSION

The Ω^- as the Result of a Dynamical Calculation

We have seen in Sec. III that the $\bar{K}\Xi$ system can be expected to have a bound or resonant state, at a mass which depends on the strength (g^2) of the single-particle-exchange contribution to the scattering amplitude. *If* $g^2\approx22$ and *if* the state has isotopic spin $T=0$, then it is identical with the recently discovered[1] member of the SU_3 decuplet representation at about 1680 MeV, the Ω^-, and may reasonably be identified with it. If, in addition, no other partial wave can be bound, then the assumptions of the present calculation would imply the existence of the Ω^-. The question of the isotopic spin of the bound or resonant state is particularly interesting, since the $\Xi\bar{K}$ state with $T=1$ can only be placed in a 27-fold representation of SU_3, and at present there is no evidence, experimental[9] or theoretical,[8] for the existence of the 27 in any spin state.

In the approximation that $m_\Lambda=m_\Sigma$, the present calculation can include both the Λ and Σ exchange in the crossed channel and give a result for the $\bar{K}\Xi T=0$ or $T=1$ state, merely by adjusting the value of g^2. We have

$$T=0:\quad g^2=3g_{K\Sigma\Xi}^2-g_{K\Lambda\Xi}^2,$$
$$T=1:\quad g^2=g_{K\Sigma\Xi}^2+g_{K\Lambda\Xi}^2.$$

That is, Σ exchange is attractive for both $T=0$ and $T=1$ while Λ exchange is attractive for $T=1$, repulsive for $T=0$. If we use the SU_3 symmetric coupling constants given by Martin and Wali,[8] we put $g_{K\Sigma\Xi}^2=15$, $0\leqslant g_{K\Lambda\Xi}^2\leqslant3.2$, so $g^2(T=0)\approx42$, $g^2(T=1)\approx18$. Then from Fig. 1 we predict a $T=0$ bound state at about 1660 MeV and a $T=1$ bound state at about 1700 MeV. If, however, we use one-half these values, we find at $T=0$ bound state at 1680 MeV and a $T=1$ resonance at 1825 MeV. The interested reader may try to sharpen these estimates.

One should also include an estimate of the effect of Y_1^* exchange (from the point of view of SU_3 this is necessary for self-consistency) and of vector-meson exchange. On the basis of analogous considerations for the πNN^* system, however, these effects are expected to perhaps be of quantitative but probably not qualitative significance.

Thus we conclude that the present calculation predicts a $J=\frac{3}{2}^+\bar{K}\Xi$ bound or resonant state for both $T=0$ and $T=1$. Because we do not know the value of g^2 we cannot make a definite prediction for the masses of these states, but for reasonable values of g^2 the $T=0$ state could have (from Fig. 1) a mass of about 1680 MeV.

It seems possible that a modified calculation, including inelastic effects, could have an effect on the g^2 versus s_R curve mainly for $s_R\gtrsim(m+\mu)^2$ (which, from Fig. 1, corresponds to small and negative g^2). That is, it would be surprising if a change in our approximate treatment of left-hand singularities had any effect on Fig. 1 for s_R above threshold. But our neglect of inelastic effects should begin to affect our results in just this region. If the pion production channel had a repulsive effect the g^2 versus s_R curve would become more rectangular. Then values like $g_{K\Sigma\Xi}^2\approx8$, $g_{K\Lambda\Xi}^2\approx0$ could give a $T=0$ bound state at about the Ω^- mass, while for $T=1$ they give $D(s)\neq0$ and therefore no $T=1$ bound or resonant state (as is the case now for $g^2\leqslant-5$).

Furthermore, suppose one imagined[10] that in the real world all forces (left-hand singularities which appear in N) are as strong as possible, meaning by this that the system is unaware of a further increase in their strength, while it would notice a decrease. One would then find it remarkable that, according to Fig. 1, such a phenomenon is (approximately) observed. This would allow us to predict the Ω^- mass (the real world) to be the

[8] A. W. Martin and K. C. Wali, Phys. Rev. **130**, 2455 (1963); Nuovo Cimento **31**, 1324 (1964).

[9] A. H. Rosenfeld, University of California Radiation Laboratory Report UCRL 10897 (unpublished).

[10] This is perhaps related to the work of G. F. Chew and S. C. Frautschi. See S. C. Frautschi, *Regge Poles and S-Matrix Theory* (W. A. Benjamin, Inc., New York, 1963), and references given there.

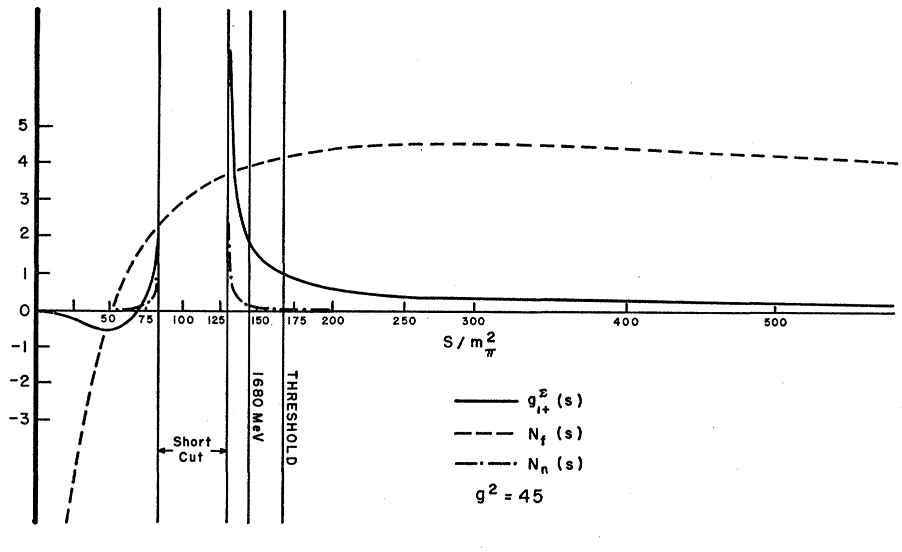

FIG. 3. $N_f(s)$ (dashed line), $N_n(s)$ (dash-dot line) and $g_{1+}^{\Sigma}(s)$ (solid line), vs. s, for $s_{m1}=75\ m_\pi^2$, $s_{m2}=135\ m_\pi^2$, $s_0=50\ m_\pi^2$, and $g^2=45$. N_n and g_{1+}^{Σ} are not shown on the short cut. Units for ordinate are m_π^{-1}, for abscissa m_π^2.

value of s_R for which the system becomes unaware of further increases in g^2, $s_R \approx (1680 \text{ MeV})^2$. The corresponding value of g^2 would, of course, be that found for the $T=0$ state in a totally self-consistent calculation.

The Effect of the Dynamical Singularities

It was noticed by Pati in his calculation that the contribution of the short cut (N_n) was apparently small compared to that of the cut for $s \leqslant 0$. That this is also the case here is clear from Table II, where g_{1+}^{Σ}, N_n,

TABLE II. Results at the matching points and at s_R for a self-consistent solution for $g^2=45$, $s_{m1}=75\ m_\pi^2$, $s_{m2}=135\ m_\pi^2$, and $s_R=137.6\ m_\pi^2$. N_n, N_f, g_{1+}^{Σ}, and g_{1+}^{Ω} are in units of m_π^{-1}, D is dimensionless, and D' is in units of m_π^{-2}.

s	g_{1+}^{Ω}	g_{1+}^{Σ}	N_n	N_f	$\text{Re}D_n$	$\text{Re}D_f$	$\text{Re}D_n'$	$\text{Re}D_f'$
s_{m1}	2.77	0.14	0.35	1.87	10^{-4}	0.24	$\cdots$	$\cdots$
s_{m2}	101.7	3.97	0.33	3.79	7×10^{-4}	0.94	$\cdots$	$\cdots$
s_R	$\cdots$	$\cdots$	0.31	3.84	0.01	0.99	10^{-4}	0.047

and D_n, all of which come from Σ exchange and are proportional to g^2, are a small fraction of g_{1+}^{Ω}, N_f, and D_f, respectively. The latter quantities depend on g^2 in a more subtle manner, from solving Eq. (23) at the matching points.

It is clear from Fig. 1, however, that our results are quite sensitive to g^2. We have not found the mechanism at the source of this apparent disagreement. The following considerations do, however, explain how our calculation (with a far left cut), and a calculation (such as that of Martin and Wali, who also find an Ω^- self-consistently) based on using for $N(s)$ the Born approximation to the amplitude, can give similar results.

As can be seen from Eq. (11), N_n contains only the short-cut part of g_{1+}^{Σ}, and as shown in Eq. (16), N_n vanishes like constant/s. If we replace N_n by g_{1+}^{Σ} and put $N_f=0$ we find two effects. First, beyond threshold g_{1+}^{Σ} is an order of magnitude larger than N_n (see Fig. 3). Second, g_{1+}^{Σ} vanishes like g^2/W (the unitarity limit on g_{1+} is $8/W$). So noting that

$$\left(\int_{(m+\mu)^2}^{\infty}\frac{ds}{s^2}\right)\Big/\left(\int_{(m+\mu)^2}^{\infty}\frac{ds}{s^{5/2}}\right)\approx 20,$$

and combining the two effects, we see that $\text{Re}D_n$ could easily increase by an amount sufficient to allow $\text{Re}D=0$ even with $\text{Re}D_f=0$.

Further, it is apparent from Fig. 3 that $N(s)$ and $g_{1+}^{\Sigma}(s)$ are quite different, so that $N(s)$ is not being self-consistently determined to equal $g_{1+}^{\Sigma}(s)$, even over some small region. But this is misleading because of the difference in asymptotic behavior. To have the same effect on D, N would, in fact, have to be several times g_{1+}^{Σ} near threshold. Thus, from the point of view of their contribution to $D(s)$, it is reasonable that N_f+N_n and g_{1+}^{Σ} can have the same effect, though their magnitudes are quite different near threshold. It is their high-energy behavior which leads to their equivalence. A physical reason for their equivalence and for the strong dependence of N_f on g^2 has not been found. We note, however, that Eq. (23), used at the matching points, does in fact determine N_f in terms of g_{1+}^{Σ} and g_{1+}^{Ω}.

The Second Zero of $D(s)$

It has been suggested by Abers and Zachariasen[4] that, in a p-wave bootstrap calculation, if $\text{Re}D(s)$ has a zero on one side of threshold it is likely to have one on the other side as well. A second zero is indeed found in the present calculation. Some examples are shown in the last part of Table I. However, the residue γ^2 at the second zero is negative. This is because $N(s)$ is positive everywhere to the right of the short cut for our calculation, while $\text{Re}D'(s)$ necessarily changes sign between the two zeros. This is equivalent to the statement that the phase shift is decreasing through $\frac{1}{2}\pi$ at the second zero, because the derivative of the phase shift is proportional to $(-\mathrm{Re}D'/N)$. Thus, our second zero is not due to the Abers-Zachariasen mechanism, and no alternative solution of the Abers-Zachariasen type is present.

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Possible Connection Between Gravitation and Fundamental Length*

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An analysis of the effect of gravitation on hypothetical experiments indicates that it is impossible to measure the position of a particle with error less than $\Delta x \gtrsim \sqrt{G} = 1.6 \times 10^{-33}$ cm, where G is the gravitational constant in natural units. A similar limitation applies to the precise synchronization of clocks. It is possible that this result may aid in the solution of the divergence problems of field theory. An equivalence is established between the postulate of a fundamental length and a postulate about gravitational field fluctuations, and it is suggested that the formulation of a fundamental length theory which does not involve gravitational effects in an important way may be impossible.

I. INTRODUCTION

THE presence of divergences in quantum field theory leads one to consider the possibility of modifying the formalism by introducing a fundamental length into the theory. Although the proof by Källén[1] has recently been questioned,[2,3] it still seems not unlikely that the renormalization constants of quantum electrodynamics and other field theories are indeed infinite. Although the renormalization theory permits one to get finite results for physically observable quantities in any order of perturbation theory, the existence of the infinite quantities makes one feel somewhat uneasy about the theory. Moreover, in the model proposed by Lee,[4] it has been shown[5] that the infinite coupling constant renormalization leads in an exact solution to the existence of physically unacceptable "ghost" states, which destroy the unitarity of the S matrix; and it may be[6] that similar difficulties are contained in the more realistic field theories as well.

It is often stated that the divergences arise from the concept of a point particle. This is true, but in a somewhat indirect sense. In the present theory, due to the possibility of pair creation, a single particle cannot be localized more closely than its Compton wave length without losing its identity as a single particle; i.e., if the mass of the particle is m, its position will be uncertain by $\Delta x \gtrsim 1/m$. (Throughout this paper we use natural units: $\hbar = c = 1$.) Therefore, it might be more accurate to say that the divergences arise from the assumption that field quantities (such as electric field strength, charge density, etc.) averaged over arbitrarily small space-time regions are observable in principle, thus making it physically meaningful to make use of local interactions in the theory. The work of Bohr and Rosenfeld[7–9] tells us how these quantities can be measured in the case of quantum electrodynamics using test bodies equipped with springs, etc. However, these authors assume that test bodies having any desired properties can exist in principle. It is clear that the average of a field quantity in a volume V cannot be measured by a test body unless the test body itself is known to be located in the volume under study. It is therefore possible to state that the divergences in a field theory arise, not from the assumption that the particles being studied in the theory are point particles, but from the assumption that point (or arbitrarily

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